15529/E 640

Reg. No. | | | | | | | | |

V Semester B.A./B.Sc. Degree Examination, November 2015

Mathematics (Optional)

(KUD – Repeaters)

Paper – I : REAL ANALYSIS

Time : 3 Hours] [Max. Marks : 80

Instruction : *Answer **all** questions.*

I. Answer **any five** of the following : **(5 × 2 = 10)**

1. Define upper and lower sums of a bounded function in $[a, b]$.

2. Find $L(P, f)$ and $U(P, f)$, if $f(x) = x$ on $[0, 1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of $[0, 1]$.

3. State First mean value theorem of integral calculus.

4. Discuss the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

5. State Abel's test for due convergence of an improper integral of product function.

6. Prove that $\Gamma n = \lfloor n-1$, $\forall\ n \in N$.

7. Evaluate $\Gamma\frac{1}{4} \cdot \Gamma\frac{3}{4}$.

8. Evaluate $\int_0^1 \int_0^3 x^2 y^3 \ dx \ dy$.

II. Answer **any eight** of the following : **(8 × 5 = 40)**

9. If $f(x)$ and $\varphi(x)$ are bounded and integrable in $[a, b]$ then prove that $f(x)+\varphi(x)$ is integrable in $[a, b]$. Also prove that

$$\int_a^b [f(x)+\varphi(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b \varphi(x)\,dx.$$

10. If $f(x)=\frac{1}{2^n}$, when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, where $n = 0, 1, 2, 3, \ldots$, $f(0)=0$.

Prove that $f(x)$ is R-integrable in [0, 1]. Also find $\int_0^1 f(x)\,dx$.

11. Prove that every monotonic function is R-integrable.

12. Prove that $\frac{1}{3\sqrt{2}} \le \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx \le \frac{1}{3}$.

13. State and prove fundamental theorem of integral calculus.

14. If $f(x)$ is bounded and R-integrable in $[a, b]$ then prove that $|f(x)|$ is also bounded and R-integrable in $[a, b]$. Give an example to show that the converse is not true.

15. If $f(x)$ and $\varphi(x)$ are positive in $[a, \infty)$ and $\lim_{x\to\infty}\left[\frac{f(x)}{\varphi(x)}\right] = l\ (\neq 0, \infty)$ then prove that the integrals $\int_0^\infty f(x)\,dx$ and $\int_0^\infty \varphi(x)\,dx$ behave alike $(a>0)$.

16. Test the convergence of :

(a) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

(b) $\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)}$.

17. Show that $\frac{\beta(p,\ q+1)}{q} = \frac{\beta(p+1,\ q)}{p} = \frac{\beta(p,\ q)}{p+1}$.

18. Prove that $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$.

19. Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$.

20. Find the volume of tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

III. Answer **any three** of the following : **(3 × 10 = 30)**

21. (a) State and prove the necessary and sufficient condition for Riemann integration.

(b) Prove that $\int_0^\pi \frac{6x^2}{1+\cos x}\,dx \geq \pi^3$.

22. (a) State and prove Weierstrass form of second mean value theorem of integral calculus.

(b) Prove that $\left|\int_p^q \frac{\sin x}{x}\,dx\right| \leq \frac{2}{p}$, where $q > p > 0$.

23. (a) State and prove Dirichlet's test for the convergence of an improper integral of product function.

(b) Show that $\int_0^\infty e^{-ax}\sin bx\,dx$ is convergent, if $a > 0$.

24. (a) Prove that $\beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma (m+n)}$.

(b) Show that $\int_0^\infty e^{-x^4}dx \times \int_0^\infty e^{-x^4}x^2\,dx = \frac{\pi}{8\sqrt{2}}$.

25. (a) With usual notation prove that $\frac{d}{dy}\int_a^b f(x, y)\,dx = \int_a^b f_y(x, y)\,dx$.

(b) Assuming the validity of differentiation under integral sign prove that $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx = \pi\left(\sqrt{1+y}-1\right)$.